A minimality property for integral models of Rapoport-Zink spaces

1

"Groupes analytiques rigides p-divisibles II"
Math. Annalen.

What is a canonical integral model?

Good notion of canonical model of Shimura variety / reflex field.

→ Find the corresponding notion for integral models (See recent work of Pappas for example)

Here: Local Shimura varieties: Rapoport-Zink spaces

Main theorem: $\mathbb{H}$ p-divisible group / $\overline{\mathbb{F}}_p$ $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{un}}$

$\mathcal{M} / \mathrm{Spf}(\breve{\mathbb{Z}}_p)$ Rapoport-Zink space of deformations of $\mathbb{H}$ by quasi-isogenies formally smooth formal scheme locally formally of finite type / $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$.

$S \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$

$$\mathcal{M}(S) = \{(H, \rho)\}/\sim$$

BT_S

quasi-isogeny

$$\rho : \mathbb{H} \otimes_{\overline{\mathbb{F}}_p} S \bmod p \longrightarrow H \times_S S \bmod p$$

Equipped with ω = vector bundle $= (\mathrm{Lie}\, H^{univ})^\vee$

~~[illegible]~~

* $\mathcal{M}_\eta$ generic fiber / $\breve{\mathbb{Q}}_p$ as a rigid space

Equipped with $\omega^+ := sp^{-1}\omega \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}^+_{\mathcal{M}_\eta}$

bounded by 1 holomorphic functions.

<u>Th</u>: $\mathfrak{X}$ formally smooth formal scheme loc. formally of finite type / $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$

$$\mathfrak{X}_\eta \xrightarrow{f} \mathcal{Y}_\eta \text{ morphism}$$

Then: f extends to $\mathfrak{X} \to \mathcal{Y}$

$$\Updownarrow$$

$f^*\omega^+$ is locally free on $|\mathfrak{X}|$ via $sp : \mathfrak{X}_\eta \to \mathfrak{X}$

$\underbrace{f^*\omega^+}_{\mathcal{O}^+_{\mathfrak{X}_\eta}\text{- module loc. free}}$

$$\Updownarrow$$

$f^*\omega^+ = \mathcal{E}^+$ where $\mathcal{E} =$ v.b. $/\mathfrak{X}$ i.e. $f^*\omega^+$ extends to $\mathfrak{X}$

<u>Rem</u>: Such an extension $\mathcal{E}$ is unique via de Jong: $\mathcal{O}_{\mathfrak{X}} = sp_* \mathcal{O}^+_{\mathfrak{X}_\eta}$

$$\Rightarrow \mathcal{E} = sp_* \mathcal{E}^+.$$

$\rightarrow$ $(\mathcal{Y}, \omega)$ is a minimal model of $(\mathcal{Y}_\eta, \omega^+)$ in the sense of birational geometry.

Ex: $\mathfrak{X}$ q.c. smooth (p-adic admissible à la Raynaud thus)

$/\mathrm{Spf}(\breve{\mathbb{Z}}_p)$

<u>Raynaud:</u> $\mathfrak{X}_\eta \xrightarrow{f} \mathfrak{Y}_\eta$ is induced by some g, $f = g_\eta$

normal

admissible formal blow-up

$$\begin{tikzcd}
\tilde{\mathfrak{X}} \arrow[d, "\pi"'] \arrow[dr, "g"] & \\
\mathfrak{X} \arrow[r, dashed, "h"] & \mathfrak{Y}
\end{tikzcd}$$

<u>Then:</u> g descends along π to h

$\Updownarrow$

$g^*\omega$ locally free on $\mathfrak{X}$ via π

$\Updownarrow$

$g^*\omega$ descends to a v.b. $/\mathfrak{X}$

Tools of the proof:

* Rigid analytic p-div. groups (I + II)
  ↑ families

* Bartenwerfer $H^1(\mathbb{B}^d_K, \mathcal{O}^+) = 0$
  ↑ closed ball

* $\omega|_{\text{Irreducible Component of } \mathcal{X}_{red}}$ is ample.

Rigid analytic p-divisible groups: $K|\mathbb{Q}_p$

G/K commutative rigid analytic group s.t.

* $\times p : G \to G$ is finite
  i.e. $G[p]$ is finite
* $\times p : G \to G$ is surjective
* $\times p : G \to G$ is top. nilpotent

usual BT. H fppf sheaf ab. gp.

* $G[p]$ finite flat
* $\times p : G \to G$ fppf épi
* $G = \bigcup_{n \geq 1} G[p^n]$ i.e. p^∞-torsion

For $G \in BT_S^{\circ}$ one can define $\left[\begin{array}{l}\mathrm{Lie}\, G = \mathcal{O}_S^{+}\text{-module} \\ \text{locally free.}\end{array}\right]$

Prop: $S = \mathbb{B}_K^n = (\underbrace{\hat{\mathbb{A}}^n_{\mathcal{O}_K}}_{p\text{-adic affine space}})_\eta$. Then

$\{\text{Formal } p\text{-divisible groups} / \hat{\mathbb{A}}^n_{\mathcal{O}_K}\} \xrightarrow{\sim} \{G \in BT_S^{\circ} \mid \mathrm{Lie}\, G \text{ is loc. free on } |\hat{\mathbb{A}}^n_{\mathcal{O}_K}|\}$

i.e. Lie G descends to a vector bundle on $\hat{\mathbb{A}}_{\mathcal{O}_K}$

→ Use Bartenwerfer.

* To prove the main theorem: use the fact that the ~~....~~ Milnor fibers of $\mathcal{X} / \mathbb{Z}_p$ formally smooth are open balls. Apply the preceding to closed balls ⊂ open balls.